Calculating the Price of Bond Convexity

No free alpha. No free convexity.

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Attribution analysis for fixed-income portfolios usually focuses on the influence of duration and convexity on the performance of a portfolio. As the convexity bias in the yield curve is well-known and accepted by portfolio managers and traders of bond and swap portfolios, how much premium should one pay to acquire convexity in a portfolio? What should the discount be if convexity is sacrificed?

The theory most frequently invoked to explain the shape of the yield curve is the expectations hypothesis (Bodie, Kane, and Marcus [1993]). The expectations hypothesis states that the forward rate for a period in the future equals the market consensus expectation of the future interest rate, and that the yield curve is therefore determined by expected future changes in interest rates.[1]

The market segmentation theory, on the other hand, states that the money market and capital market section of the yield curve are independent, and that the yield curve is determined by the equilibrium rates set in the different markets. McCulley [2004] discusses one version of this theory, arguing that rates in the bond market are set independently from short-term rates.

Phoa [1997] introduces another theory for explaining the shape of the yield curve, also called *expectations theory*. He argues that the short rate and the long rate could be seen as being joined by an instantaneous forward rate curve. He also derives a formula for the forward rate curve.

Our aim is to go beyond these theories. While the shape of the yield curve is traditionally explained by the expectations hypothesis, the convexity bias in the yield curve is well known. Fabozzi [1993] notes that the market prices convexity, and that the cost of the convexity is the yield give-up. The price depends on investors' expectations about the volatility of future interest rates—the greater the expected volatility, the more an investor should be willing to pay for convexity.

Rebonato [1997] also illustrates that, given the same expectations of future interest rates, different expected future volatilities produce different discount bond prices. Litterman, Scheinkman, and Weiss [1991] also examine the role of interest rate volatility in determining the shape of the yield curve. They remark that: "Understanding the effect of the interest rate volatility implicit in the yield curve is essential to comprehending the behavior of fixed-income securities" [1991, p.].

INFLUENCE OF VOLATILITY ON A BOND PRICE

As time passes and interest rates change, the price of a bond changes. Chance and Jordan [1996] show that the time dimension should be included in the expansion of the change in bond price P for a particular change in the yield, z. They propose the Chance-Jordan decomposition for bond returns R over interval $[t, t + \Delta t]$:

$$R = \Delta P / P$$

where

$$\Delta P(z,t) \approx \frac{dP}{dz}\Delta z + \frac{dP}{dt}\Delta t + \frac{1}{2}\frac{d^2P}{dz^2}\Delta z^2 + \frac{d^2P}{dzdt}\Delta z \Delta t + \frac{1}{2}\frac{d^2P}{dt^2}\Delta t^2$$

This equation is a Taylor expansion, truncated after five terms. The first term is the duration term. Chance and Jordan show that the theta term $(dP/dt)\Delta t$ can be far more important than the convexity term $(1/2)(d^2P/dz^2)\Delta z^2$. While the contribution of the last term is negligible, the interaction term $(d^2P/dzdt)\Delta z\Delta t$ can also be important. If $\Delta z = 0$, the return can be approximated by the theta term only, assuming that the last term is negligible.

We build on these various insights and derive our expected volatility hypothesis using a hedged bond portfolio. We also derive a formula for constructing a yield curve.

HEDGING PROBLEM

In practical terms, what are the implications of these convexity, volatility, and theta issues for pension funds and hedge funds?

The global decline in interest rates means many pension funds have experienced difficulties with deficits building up to billions of dollars. Most defined-benefit pension funds must hedge future liabilities with available assets in the bond market in order to eliminate interest rate risk. If interest rates move, the value of the pension fund's assets must always equal the value of the fund's liabilities in order to remain funded. Therefore, the change in the value of the assets must equal the change in the value of the liabilities over a short period of time.

To quantify this effect at time t, we approximate the change in the present value f of any future liability or cash flow at time t_1 in terms of the truncated Taylor series (Chance-Jordan decomposition):

$$\Delta f(z(t_1 - t), t) = \frac{df}{dz}\Delta z + \frac{df}{dt}\Delta t + \frac{1}{2}\frac{d^2 f}{dz^2}\Delta z^2 + \frac{d^2 f}{dzdt}\Delta z \Delta t \qquad (1)$$

where $z(t_1 - t)$ represents the yield for an investment with $(t_1 - t)$ time to maturity, and Δz represents the change in yield over one time interval Δt. For our purposes, we assume that $\Delta z \neq 0$, and that Δz is approximated by an average historic daily volatility over a one-year period. The second-order time derivative has been discarded, as suggested by the empirical observations of Chance and Jordan [1996].

The first term in the equation (the duration term) is usually used to give an approximate hedge for a portfolio (Barber [1995]). For example, a defined-benefit pension fund's liabilities are hedged mainly by matching the duration of the liabilities with assets with a similar duration and profile. A duration hedge is usually effective for short-term liabilities, but the convexity term [the third term in the Taylor series, Equation (1)] starts to play a more significant role as the term to maturity increases (Kang and Chen [2002]). The main problem with hedging the liabilities with shorter-term assets (by borrowing) is that one will be duration-hedged, but will still be exposed to convexity risk.

Hedge funds use long-short positions in bonds to profit from the difference in the yield to maturity for bonds. Long-Term Capital Management noticed in 1994 that the 29.5-year U.S. Treasury bond, trading at 6.44% yield, seemed cheap compared to the 30.0-year Treasury bond, which was trading at a yield of 6.33%, partly because of a lack of liquidity. LTCM went long the 29.5-year bond and short the 30.0-year bond. Although it profited at the time from this trade, was this spread between the bonds at fair value, and if not, what should it be?

Although liquidity would play a role in these spreads, we argue that the convexity effect becomes very important for long-term bonds and therefore has a major influence in construction of a portfolio. Assuming permanent market segmentation between the money market and the bond market, and that the expectations hypothesis holds for the money market sector of the yield curve, we propose a hypothesis for the term structure of interest rates.

Expected Volatility Hypothesis

We propose that the yield to maturity of any point on the capital market section of the yield curve is influenced by the no-arbitrage relationship between convexity and theta, and the convexity-theta interaction, for different duration-hedged bonds, given a certain expected volatility, the short rate, and one benchmark bond rate.[2]

Convexity Effect

The price (present value) at time t of a zero-coupon bond paying 1 unit at time t_1 is given by:

$$f(z(t_1 - t), t_1 - t) = \exp[-z(t_1 - t)(t_1 - t)] \tag{2}$$

where $z(T)$ = continuously compounded zero-coupon yield for a T-term investment.

It is well known that the function is more convex when $(t_1 - t)$ is large. As Fabozzi [1993] and Barber and Copper [1997] note, convexity does not provide a free lunch, and it should be consistent with arbitrage-free pricing. Therefore, we propose a term structure model that prices the arbitrage-free yield curve, using our expected volatility hypothesis.

TERM STRUCTURE MODEL FOR EXPECTED VOLATILITY HYPOTHESIS

Assume that S is a portfolio consisting of two different maturity Treasury zero-coupon bonds (one long and one short position) as well as a cash amount c_0 at time $t = t_0$. The portfolio is hedged so that at time $t = t_0$ the value of the portfolio is zero (self-financing) and the two bonds are duration-hedged.

Let f_1 and f_2 denote the value of the short and long positions, respectively, with the nominal amounts of the short and long bond one and N, respectively. The values of the two positions at time t are given by:

$$f_1(t) = -\exp[-z(t_1 - t)(t_1 - t)] \tag{3}$$

$$f_2(t) = N\exp[-z(t_2 - t)(t_2 - t)] + c_0 e^{r(t - t0)} \tag{4}$$

where f_2 includes a cash amount, c_0 invested/borrowed at the short rate $r = z(\Delta t)$ at the initial time t_0.

From our assumptions, we know that at time $t = t_0$:

$$S = f_1 + f_2 = 0 \tag{5}$$

$$\text{and} \quad \frac{dS}{dz} = 0 \tag{6}$$

The no-arbitrage principle would argue that this portfolio should not make risk-free profits over a short period of time Δt (say, one day).

Therefore, at time t_0, using Equation (1), the approximate values of the *change* in value for the two positions over a time interval Δt are given by:

$$\Delta f_1(t_0) = -f_1[(t_1 - t_0)\Delta z - z(t_1 - t_0)\Delta t - \frac{1}{2}(t_1 - t_0)^2 \Delta z^2 - [1 - z(t_1 - t_0)(t_1 - t_0)]\Delta z \Delta t] \tag{7}$$

and

$$\begin{aligned} \Delta f_2(t_0) = {} & -(t_2 - t_0)(f_2 - c_0)\Delta z + \\ & [rc_0 + z(t_2 - t_0)(f_2 - c_0)]\Delta t + \\ & \frac{1}{2}(t_2 - t_0)^2 (f_2 - c_0)\Delta z^2 + \\ & [f_2 - z(t_2 - t_0)(t_2 - t)(f_2 - c_0)]\Delta z \Delta t \end{aligned} \tag{8}$$

For the portfolio to be duration-neutral at $t = t_0$, it must satisfy:

$$\frac{df_1}{dz} = -\frac{df_2}{dz} \tag{9}$$

Therefore, one can solve for the amount c_0 that must be borrowed/lent at time $t = t_0$:

$$c_0 = \frac{(t_1 - t_2) f_1(t_0)}{t_2 - t_0} \tag{10}$$

for $t_2 > t_0$. Since $f_1 < 0$, we see that, if $t_2 > t_1$, c_0 is positive (cash lent), and when $t_2 < t_1$, it is negative (money borrowed). One can then also solve for the nominal amount, N in Equation (4):

$$N = -f_1(t_0)\frac{t_1 - t_0}{t_2 - t_0} e^{z(t_2 - t_0)(t_2 - t_0)}$$

Going back to the Taylor series approximation, we have eliminated the duration terms for both f_1 and f_2 (being equal), and the remaining terms are the convexity term, the theta term, and the interaction term. This brings us to the no-arbitrage relationship.

The basis for the expected volatility hypothesis is the assumption that the no-arbitrage relationship for convexity, theta, and the interaction between them holds. Litterman, Scheinkman, and Weiss [1991] call this the *one-period rate of return expectations hypothesis*.

Litterman and Scheinkman [1991] show that different capital market instruments are highly correlated. Furthermore, the convexity adjustment corresponding to the non-parallel shifts in the yield curve is negligible. We can argue accordingly from the principle of no-arbitrage that all *duration-hedged* bonds (with similar credit risk) across the yield curve should have the same change in price overnight for a certain move in the yield curve (see Lacey and Nawalkha [1995] and Barber and Copper [1997]). We thus assume that no duration-hedged long-short Treasury bond position should make risk-free profits overnight, which implies that $-\Delta f_1$ should equal Δf_2.

Let $T_2 = t_2 - t_0$ and $T_1 = t_1 - t_0$ be the time to maturity for the two different duration-hedged bonds at time $t = t_0$. Then, from the remaining terms in Equations (7) and (8), it can be shown that the term structure at time $t = t_0$ is given by:

$$z(T_2) = \frac{[z(T_1)f_1(t_0) - rc_0]\Delta t}{(1 - T_2\Delta z)(f_1(t_0) - c_0)\Delta t} + \frac{\frac{1}{2}[(T_1^2 - T_2^2)f_1(t_0) + T_2^2 c_0]\Delta z^2}{(1 - T_2\Delta z)(f_1(t_0) - c_0)\Delta t} - \frac{f_1(t_0)z(T_1)T_1\Delta z\Delta t}{(1 - T_2\Delta z)(f_1(t_0) - c_0)\Delta t} \tag{11}$$

or, by substituting Equation (10), we can eliminate f_1 and c_0 to give:

$$z(T_2) = \frac{z(T_1)[T_2(1 - T_1\Delta z)] - r(T_2 - T_1)}{(1 - T_2\Delta z)T_1} + \frac{1}{2}\frac{[T_1^2 T_2 - T_2^2 T_1]\Delta z^2}{(1 - T_2\Delta z)T_1\Delta t} \tag{12}$$

Assuming a non-zero value for volatility (or expected volatility) Δz in a time interval Δt, which gives the expected one-day parallel movement in $z(T)$, we can solve for the value of $z(T_2)$, given $z(T_1)$ as the benchmark rate. This means we have a closed-form formula to construct the yield curve.

EMPIRICAL RESULTS

We need three parameters to obtain a theoretical yield curve when we apply the expected volatility hypothesis to actual data:

- Expected volatility Δz.
- Short rate r.
- One benchmark bond rate $z(T_1)$.

The short rate can be assumed to be the overnight deposit rate, while the benchmark rate can be any benchmark government bond on the yield curve. Let us assume the benchmark rate is the five-year government zero-coupon bond yield.

The only unknown parameter is the expected volatility. This can be approximated by, for example, the historical volatility or implied bond option volatility. Phoa [1997] notes several problems with the use of implied volatility. Implied volatility given by swaptions would be comparable, but this is not always quoted. One can also use the actual yield curve to obtain the *implicit volatility* through our closed-form solution in (11) or (12), unlike Phoa [1997], who aimed to obtain this empirically. Litterman, Scheinkman, and Weiss [1991] also apply a different form of regression analysis in their attempt to measure implied volatility.

We apply the expected volatility hypothesis to various countries with yield curves out to 30 years. We show actual market data (from STRIPS, as given in Exhibit 1), against the calculated best fit for the yield curve, which gives the implicit volatility in Exhibit 2 and the U.S. yield curve in Exhibit 3. The actual short rate and the five-year rate for each country are used in Equation (12) in order to calculate the curve.[3]

EXHIBIT 1
Yield Curve Rates (semiannual) Given by STRIPS (February 28, 2005)

Term	US	Canada	UK	Japan	Italy	Germany	South Africa
1	3.22	2.77	4.71	0.02	2.27	2.25	7.28
2	3.57	2.96	4.76	0.11	2.51	2.52	7.19
5	4.03	3.82	4.76	0.62	3.16	3.16	7.78
10	4.60	4.53	4.75	1.51	3.90	3.81	7.88
15	4.91	4.86	4.69	1.84	4.23	4.10	7.50
20	4.99	5.01	4.61	2.20	4.45	4.26	6.99
30	5.00	4.99	4.44	2.64	4.55	4.34	–

Source: Bloomberg.

EXHIBIT 2
Summary of Historical Volatility Against Implicit Volatility

Country	Latest 90-day historical volatility of 5-year rate	Average historical volatility of 5-year rate	Implicit volatility from term structure model	Error relative to actual rates
US	31.8	20.2	22.6	3.5
Canada	25.4	20.0	28.1	8.3
UK	13.9	11.3	8.9	0.3
Japan	65.9	41.5	68.0	1.8
Italy	25.2	20.5	28.1	2.0
Germany	22.3	19.7	28.0	2.7
South Africa	12.3	11.9	15.0	0.4

Source: INet-Bridge.

Influence of Increase in Volatility

Because we are able to price the influence of expected volatility on long-dated bonds, we can now again consider the LTCM relative-value trade between the 29.5-year bond and the 30.0-year bond. In the case of expected low volatility, this would be a very good trade, and would profit as long as volatility remains low. If volatility increases, however (as was the case in 1997–1998), the curve would become steeper at the far end if the market behaved according to the expected volatility hypothesis as shown in Exhibit 4—and this trade would lose money. For example, for a volatility of 23%, the theoretical spread should be +2 basis points, while for a volatility of 42% it changes to −30 basis points.

In a similar way, our theory can be used to determine overvalued and undervalued bonds in the market, relative to a certain expected volatility.

EXHIBIT 3
U.S. Zero-Coupon Yield Curve (February 2005)

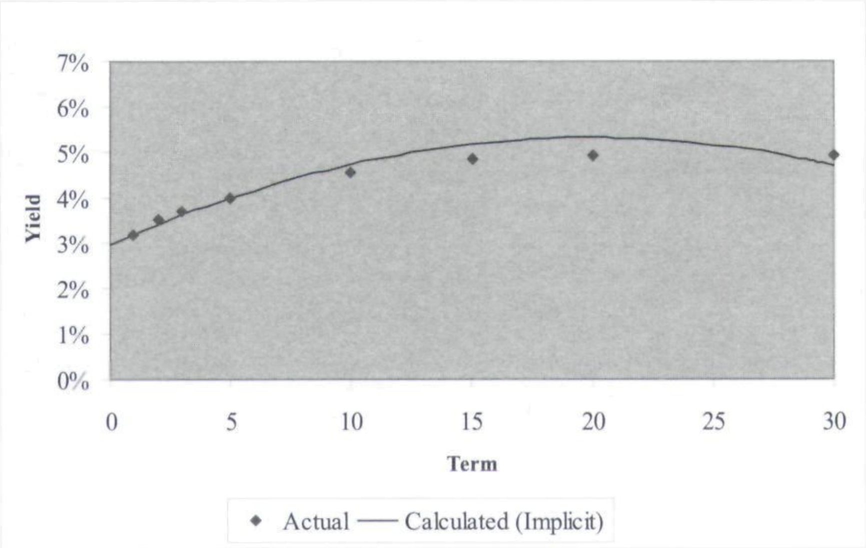
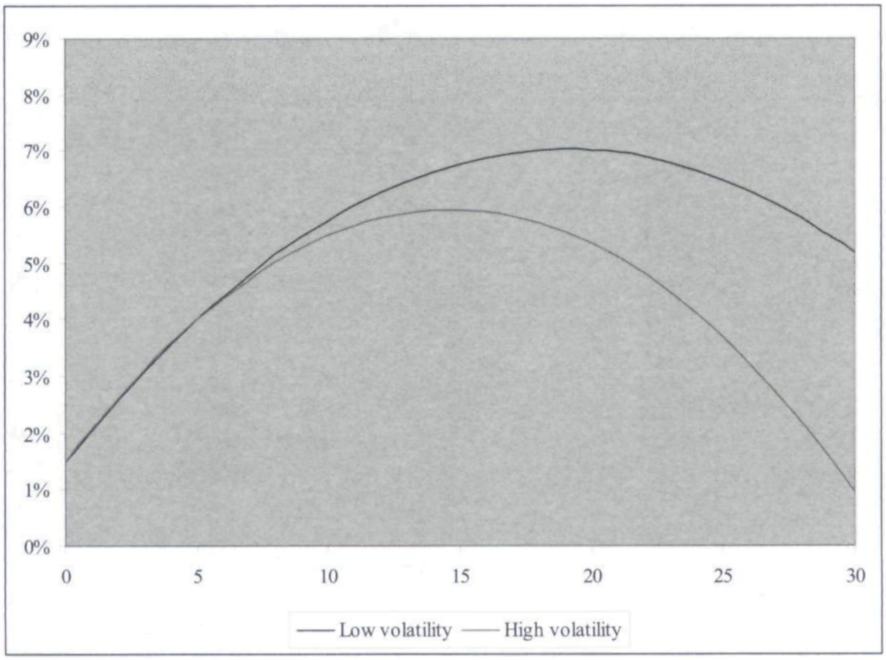


EXHIBIT 4
Influence of Increase in Expected Volatility



Limitations and Restrictions

Our model has certain restrictions in the case of a negative sloping yield curve (when there is the expectation of a decline in short rates). In order to ensure that all yields on the implied yield curve remain positive, $z(t_2)$ should always be positive for up to 30 years. We therefore introduce conditions for the volatility in relation to the actual rates in the market, given a *five-year benchmark* rate and a Δt of one day:

$$\Delta z < \sqrt{\frac{z(5) - \frac{5}{6}r}{22812.5}}$$

and

$$z(5) > \frac{5}{6}r$$

Although we argue that the shape of the yield curve can be explained mainly by the expected volatility hypothesis, there can be other influences that can change the shape of the yield curve temporarily. For example, additional market segmentation between intermediate and long-term bonds can occur when there is high demand for longer-term bonds and limited supply. The monetary policy of a country could also influence the shape of the yield curve.

One could argue that uncertainty regarding expected inflation and growth would influence not only the benchmark bond rate but also the market expectation of future volatility, and therefore the shape of the yield curve. On the other hand, the shape of the yield curve might forecast a recession, where a negative-shaped yield curve in the U.S. has indicated an upcoming recession.

CONCLUSION

The term structure model we develop opens the door to formalizing arbitrage-free pricing between convexity and theta in a fixed-income portfolio.

The main benefit of the expected volatility hypothesis is that it helps us to understand the shape of the yield curve better, and to be able to calculate the benefit of convexity in the hedging of an asset-liability managed fund and also understand the risks better. This theory is even more important in emerging market countries—where rates are relatively high and the convexity impact even greater. This would also explain why the longest-term bond in South Africa (R186) has always been traded at lower yields than shorter bonds. Over the last few years, as the term to maturity has become shorter, this effect has diminished.

Our results should help risk managers calculate the risk of long-short positions in bonds for hedge funds more accurately. As in the gamma-theta trade-off in managing an options book, it is also important to assess the convexity-theta trade-off in bonds and swaps.

The most important contribution of our model is our theory that the shape and the steepness of the *bond market* sector of the yield curve are determined mainly by the expectation of volatility for longer-term rates and not expected future rates. We would still argue that the expected movement of interest rates influences the *money market sector* of the yield curve, and that there is market segmentation between the money market and the bond market.

We assume the expected change in future rates is reflected in the benchmark bond rate. Yet the level of the 20-year bond yield does not so much reflect the expectation of future rate movements, but rather the expected volatility. A certain 5-year benchmark rate of 4% and a 20-year yield of 9% indicate a relatively low volatility expectation, rather than a higher yield expectation for the next 20 years. Similarly, a 20-year bond yield of 6% would not imply a 20-year view of low interest rates, but rather higher expected volatility.

ENDNOTES

The authors thank Wesley Phoa for his valuable comments.

[1]Yield curve refers to the zero-coupon yield curve.

[2]We assume the expectations hypothesis influences the level of the benchmark bond rate.

[3]The value for Δz is equivalent to a one-day movement, while the volatility given in the document is an annualized figure.

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